Linear systems and determinantal random point fields

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Abstract

Tracy and Widom showed that fundamentally important kernels in random matrix theory arise from systems of differential equations with rational coefficients. More generally, this paper considers symmetric Hamiltonian systems and determines the properties of kernels that arise from them. The inverse spectral problem for self-adjoint Hankel operators gives sufficient conditions for a self-adjoint operator to be the Hankel operator on $L^2(0,\infty)$ from a linear system in continuous time; thus this paper expresses certain kernels as squares of Hankel operators. For suitable linear systems $(-A,B,C)$ with one dimensional input and output spaces, there exists a Hankel operator Γ with kernel $\phi_{(x)}(s+t) = Ce^{-(2x+s+t)A}B$ such that $g_x(z) = \det(I+(z-1)\Gamma\Gamma^\dagger)$ is the generating function of a determinantal random point field on $(0,\infty)$. The inverse scattering transform for the Zakharov–Shabat system involves a Gelfand–Levitan integral equation such that the trace of the diagonal of the solution gives $\frac{\partial}{\partial x}\log g_x(z)$. Some determinantal point fields in random matrix theory satisfy similar results.

Keywords: Determinantal point processes; random matrices; inverse scattering

1. Introduction

Traditionally, one begins random matrix theory by defining families of self-adjoint $n\times n$ matrices endowed with probability measures, known as ensembles, and then one determines the joint distribution of the random eigenvalues. By scaling the variables and letting $n\to\infty$, one obtains various kernels which reflect the properties of large random matrices. The kernels generate determinantal random point fields in Soshnikov's sense [16, 20]. It turns out that many such kernels in random matrix theory have the form

$$K(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y} \qquad (x,y>0) \tag{1.1}$$

where f and g satisfy the system of differential equations

$$m(x)\frac{d}{dx}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix} = \begin{bmatrix}\alpha(x) & \beta(x)\\ -\gamma(x) & -\alpha(x)\end{bmatrix}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix}, \tag{1.2}$$

where m,α,β and γ are real polynomials. Tracy and Widom [17, 18, 19] began what amounts to a classification of kernels that arise from such differential equations, and their analysis revealed detailed results about the fundamental ensembles.

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Of particular interest is the Airy kernel

$$K_\lambda(x,y) = \frac{\mathrm{Ai}(x-\lambda)\mathrm{Ai}'(y-\lambda) - \mathrm{Ai}'(x-\lambda)\mathrm{Ai}(y-\lambda)}{x-y} \tag{1.3}$$

on $L^2(0,\infty)$ where Airy's function Ai satisfies $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$. Some of the fundamental properties of this kernel involve the remarkable formula

$$K_\lambda(x,y) = \int_0^\infty \mathrm{Ai}(x+u-\lambda)\mathrm{Ai}(u+y-\lambda)\,du \tag{1.4}$$

which expresses the operator K as the square of the Hankel operator on $L^2(0,\infty)$ that has kernel $\mathrm{Ai}(x+y-\lambda)$.

The differential equation (1.2) is an example of a symmetric Hamiltonian system, as we can define more generally.

Definition *(Symmetric Hamiltonian system).* For an integer $m \geq 1$, let J be the matrix

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}, \tag{1.5}$$

which satisfies $J^2 = -I_{2m}$ and $J^T = -J$, and let $E(x)$ and $F(x)$ be $(2m)\times(2m)$ real symmetric matrices for each $x > 0$ such that $x \mapsto E(x)$ and $x \mapsto F(x)$ are continuous. Then we consider the symmetric Hamiltonian system

$$J\frac{d}{dx}\Psi_\lambda = \big(\lambda E(x) + F(x)\big)\Psi_\lambda \tag{1.6}$$

where $\Psi_\lambda(x)$ is a $(2m)\times 1$ complex vector. In particular, when $E(x)$ and $F(x)$ have entries that are rational functions of x, we have a system considered by Tracy and Widom [19].

Given a solution $\Psi_\lambda \in L^\infty((0,\infty);\mathbf{C}^{2m})$, we introduce the kernel

$$K_{s,\lambda}(x,y) = \frac{\langle J\Psi_\lambda(x+s), \Psi_\lambda(y+s)\rangle_{\mathbf{R}^{2m}}}{x-y} \qquad (x,y>0) \tag{1.7}$$

and we investigate the properties of $K_{s,\lambda}$.

More generally, we consider families of kernels $K_{t,\lambda}(x,y)$ for $t,\lambda > 0$, that satisfy some of the following properties as operators on $H = L^2(0,\infty)$:

(1^o) the Lyapunov equation holds

$$\frac{\partial}{\partial s}K_{s,\lambda} = -AK_{s,\lambda} - K_{s,\lambda}A^\dagger \tag{1.8}$$

as a sesquilinear form on $D(A^\dagger)\times D(A^\dagger)$, where e^{-sA} is a bounded C_0 semigroup on H and $D(A^\dagger)$ is the domain of $A^\dagger$;

(2^o) $0 \leq K_{s,\lambda} \leq I$ for all $s \geq s_0$ for some $s_0 < \infty$;

(3^o) $s \mapsto K_{s,\lambda}$ is decreasing and converges strongly to 0 as $s \to \infty$;

(4^o) $K_{s,\lambda}$ is of trace class;
(5^o) the operator on H with kernel $\frac{\partial}{\partial s}K_{s,\lambda}$ has finite rank.

In fact, many of the properties of ensembles which arise in random matrix theory are essentially consequences of the properties $(1^o)-(5^0)$, in a sense which we now make more precise. We recall from [16] the notion of a determinantal random point field.

Definition *(Configurations).* A configuration on $\mathbf{R}$ is an ordered list $\lambda=(\lambda_j)_{j=-\infty}^{\infty}$ such that $\lambda_j\leq\lambda_{j+1}$ for all $j\in\mathbf{Z}$; the configuration is locally finite if $\nu_\lambda(L)=\sharp\{j:\lambda_j\in L\}$ is finite for all compact sets L. Let Λ be the space of all locally finite configurations on $\mathbf{R}$. For each bounded and Borel set E, and $k=0,1,\dots,$ we let

$$C_k^E=\{\lambda\in\Lambda:\nu_\lambda(E)=k\}$$

be the set of all locally finite configurations that have k terms in E; now let B be the σ-algebra generated by the C_k^E. A random point field $(\mathbf{P},\Lambda,B)$ on $\mathbf{R}$ is a probability measure $\mathbf{P}:B\to[0,1]$. We let $\nu(a,b)$ be the random variable that gives the number of points in (a,b), so $\nu(x,\infty)=\sharp\{j:\lambda_j>x\}$.

Definition *(Correlation functions).* Given nonnegative integers n_j such that $\sum_{j=1}^k n_j=n$ and disjoint Borel sets E_j we consider $\lambda\in\Lambda$ such that $\nu_\lambda(E_j)\geq n_j$ for all j. Then

$$N_{E_j,n_j;j=1,\dots,k}=\prod_{j=1}^{k}\frac{\nu_\lambda(E_j)!}{(\nu_\lambda(E_j)-n_j)!}\tag{1.9}$$

gives the number of ways of choosing n_j points λ_ℓ from the $\nu_\lambda(E_j)$ points of λ that are in E_j for all j. The correlation function $R_n:\mathbf{R}^n\to\mathbf{R}_+$ for $\mathbf{P}$ is a locally integrable function, which is symmetrical with respect to permutation of its variables, such that

$$\mathbf{E}N_{E_j^{n_j};j=1,\dots,k}=\int_{E_1^{n_1}}\dots\int_{E_k^{n_k}}R_n(x_1,\dots,x_n)dx_1\dots dx_n\tag{1.10}$$

for all disjoint Borel sets E_j $(j=1,\dots,k)$. This is the expected number of configurations that have $\nu_\lambda(E_j)\geq n_k$ for all j.

Conversely, Soshnikov [16] observed that one can introduce a random point field from the determinants of kernels that satisfy minimal conditions. We state without proof the following general existence theorem for determinantal random point fields.

Lemma 1.1. *Suppose that $K:\mathbf{R}\times\mathbf{R}\to\mathbf{C}$ is a continuous kernel such that*
(1^o) the integral operator with kernel $K(x,y)$ on $L^2(\mathbf{R})$ satisfies $0\leq K\leq I$;
(4^o) the kernel $\mathbf{I}_{[a,b]}(x)K(x,y)\mathbf{I}_{[a,b]}(y)$ on $L^2(\mathbf{R})$ is of trace class for all finite a,b.
Then there exists a random point field such that the correlation functions satisfy

$$R_n(x_1,\dots,x_n)=\det\big[K(x_j,x_k)\big]_{j,k=1}^{n}\qquad(n=1,2,\dots).\tag{1.11}$$

Further, the R_n $(n=1,2,\dots)$ uniquely determine $\mathbf{P}$.

Definition *(Determinantal point field).* If the R_n have the form (1.11), then $(\mathbf{P}, \Lambda, B)$ is a determinantal random point field.

In this paper we introduce natural examples of kernels K by means of linear systems, and recover properties $(1^o)-(5^o)$ in a systematic manner. We summarise the construction in this introduction, and describe the details in section 2. Consider an operator A with domain $D(A)$ in state space H such that the C_0 semigroup e^{-tA} is bounded, so $\|e^{-tA}\| \leq M$ for some $M < \infty$ and all $t > 0$. Consider the linear system

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \qquad (X(0) = 0) \\ Y &= CX \end{aligned} \tag{1.12}$$

where $B : \mathbf{C} \to D(A)$ is bounded and $C : D(A) \to \mathbf{C}$ is admissible for $A^\dagger$, so $Y \in L^2(0, \infty)$. We let $\phi(x) = Ce^{-xA}B$ and $\phi_{(x)}(y) = \phi(y + 2x)$, then introduce the Hankel operators

$$\Gamma_{\phi_{(x)}} f(y) = \int_0^\infty \phi_{(x)}(y + u) f(u)\, du \tag{1.13}$$

from a suitable domain in $L^2(0, \infty)$ into $L^2(0, \infty)$. We also consider the Gelfand–Levitan integral equation

$$T(x, y) - \Phi(x + y) - \int_x^\infty T(x, z)\Phi(z + y)\, dz = 0 \qquad (0 < x \leq y < \infty) \tag{1.14}$$

where S and Φ are both either (i) real scalars, (ii) 2×2 real diagonal matrices, or (iii) 2×2 complex matrices. We state our main theorem as follows.

Theorem 1.2. *Suppose that the controllability Gramian*

$$L_x = \int_x^\infty e^{-tA} B B^\dagger e^{-tA^\dagger}\, dt \qquad (x > 0) \tag{1.15}$$

is of trace class on H and of operator norm $\|L_x\| < 1$; likewise suppose that the observability Gramian

$$Q_x = \int_x^\infty e^{-tA^\dagger} C^\dagger C e^{-tA}\, dt \qquad (x > 0) \tag{1.16}$$

is of trace class on H and that $\|Q_x\| < 1$.
(i) If $C = B^\dagger$ and $A = A^\dagger$, then let $g_x(z) = \det(I + (z - 1)\Gamma_{\phi_{(x)}})$ and $\Phi(x) = \phi(x)$.
(ii) If $\phi(x)$ is real, then let $g_x(z) = \det(I + (z - 1)\Gamma^2_{\phi_{(x)}})$ and $\Phi(x) = \mathrm{diag}[-\phi(x), \phi(x)]$.
(iii) Or let $g_x(z) = \det(I + (z - 1)\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}})$ and

$$\Phi(x) = \begin{bmatrix} 0 & \bar{\phi}(x) \\ -\phi(x) & 0 \end{bmatrix} \qquad (x > 0).$$

Then in each case there exists a determinantal random point field on $(0,\infty)$ *with generating function* $g_x(z)=\mathbf{E}z^{\nu(x,\infty)}$ *such that*

$$\frac{\partial}{\partial x}\log g_x(0)=\operatorname{trace}T(x,x)\qquad (x>0)\tag{1.17}$$

is given by the diagonal of the solution of the Gelfand–Levitan integral equation (1.14).

The integral equation in case (i) is associated with the inverse scattering problem for the Schrödinger equation on the real line and in case (ii) by a pair of Schrödinger equations; whereas the integral equation in (iii) is associated with a Zakharov–Shabat system.
The fundamental examples of determinantal random point fields in random matrix theory involve kernels associated with self-adjoint Hamiltonian systems of differential equations. In section 3 we introduce the notion of a symmetric Hamiltonian system with matrix potential, as considered previously by Atkinson and many others; see [5]. We consider spatial kernels K_λ associated with symmetric Hamiltonian systems, and give a sufficient condition for the kernel to factor as $K_\lambda=\Gamma_\lambda^\dagger\Gamma_\lambda$, where Γ_λ is a vectorial Hankel operator. As we show in section 4, this covers some fundamental examples of kernels that arise in random matrix theory, and we recover case (ii) of Theorem 1.2. A similar computation shows how (iii) arises.

Schrödinger differential operators on $L^2(0,\infty)$ with bounded potentials give rise to kernels in the spectral variables which satisfy (5^o), as we discuss in section 5. The Korteweg–de Vries flow has a natural effect on the kernels. In section 6, we consider the Zakharov–Shabat system and establish case (iii) of Theorem 1.2; here the kernels behave naturally under the flow associated with the cubic nonlinear Schrödinger equation. Some of the calculations will be familiar to specialists in the theory of scattering from [1, 6, 23], but we include them here so that the paper is self-contained.

2. Linear systems and their determinants

Definition *(Linear system).* Let H be a separable complex Hilbert space, called the state space, and H_0 a separable complex Hilbert space called the output space. Let e^{-sA} be a C_0 semigroup on H, such that $\|e^{-sA}\|\leq M$ for some $M<\infty$ and all $s>0$, and let $D(A)$ be the domain of the generator $-A$, which is a dense linear subspace of H and itself a Hilbert space for the norm $\|\xi\|_{D(A)}=\big(\|\xi\|_H^2+\|A\xi\|_H^2\big)^{1/2}$. In the language of linear systems from [14, 15], we consider the continuous time system

$$\begin{aligned}\frac{dX}{dt}&=-AX+BU\qquad (t>0)\\ Y&=CX,\qquad X(0)=0.\end{aligned}\tag{2.1}$$

where $B:H_0\to D(A)$ and $C:D(A)\to H_0$ are bounded linear operators; this is known as a $(-A,B,C)$ system. Let $\phi(x)=Ce^{-Ax}B$, so $\phi\in L^\infty((0,\infty);\mathrm{B}(H_0))$. The associated Hankel operator Γ_ϕ is the integral operator

$$\Gamma_\phi f(x)=\int_0^\infty \phi(x+y)f(y)\,dy,\tag{2.2}$$

defined from some dense linear subspace of $L^2((0,\infty);H_0)$ into $L^2((0,\infty);H_0)$.

We introduce the transfer function

$$\hat\phi(\lambda)=C(\lambda I+A)^{-1}B, \tag{2.3}$$

which we recognise as the Laplace transform of $\phi(x)=Ce^{-Ax}B$; the Fourier transform of ϕ gives the scattering data. Suppose that $U\in L^2((0,\infty);H)$ has Laplace transform $U(\lambda)$, and that $\hat\phi:\mathbf{C}_+\to \mathrm{B}(H_0)$ is a bounded analytic function. Then $Y\in L^2((0,\infty);H_0)$ has Laplace transform $\hat Y$ such that $\hat Y(\lambda)=\hat\phi(\lambda)\hat U(\lambda)$.

Definition *(Admissible).* We say that a bounded linear operator $C:D(A)\to H_0$ is admissible for e^{-sA} if $Ce^{-sA}\xi$ belongs to $L^2((0,\infty);H_0)$ for all $\xi\in H$, and there exists $K_C(A)$ such that

$$\int_0^\infty \|Ce^{-sA}\xi\|^2_{H_0}\,ds\le K_C(A)^2\|\xi\|^2_H\qquad (\xi\in H), \tag{2.4}$$

equivalently, the operator $\Theta^\dagger:H\to L^2((0,\infty);H_0)$ is bounded where $\Theta^\dagger\xi=Ce^{-sA}\xi$ and $\|\Theta\|=K_C(A)$. Examples in [9] show that the notion of admissibility is difficult to characterize simply.

Definition *(Schatten ideals).* Let c^2 be the space of Hilbert–Schmidt operators, and c^1 be the space of trace class operators, with the usual norms.

Proposition 2.1. *Suppose that C is admissible for e^{-sA} and that $B:H_0\to D(A^\dagger)$ has $B^\dagger$ admissible for $e^{-sA^\dagger}$. Then the observability Gramian*

$$Q_x=\int_x^\infty e^{-sA^\dagger}C^\dagger Ce^{-sA}\,ds\qquad (x>0) \tag{2.5}$$

and the controllability Gramian

$$L_x=\int_x^\infty e^{-sA}BB^\dagger e^{-sA^\dagger}\,ds\qquad (x>0) \tag{2.6}$$

define bounded linear operators $Q_x,L_x:H\to H$ by these strongly convergent integrals such that:
(1^o) the derivatives satisfy the Lyapunov equations

$$\frac{\partial Q_x}{\partial x}=-A^\dagger Q_x-Q_xA,\quad \frac{\partial L_x}{\partial x}=-AL_x-L_xA^\dagger \tag{2.7}$$

as sesquilinear forms on $D(A)\times D(A)$ and $D(A^\dagger)\times D(A^\dagger)$ respectively;
(2^0) $0\le Q_x\le K_C(A)^2I$ and $0\le L_x\le K_{B^\dagger}(A^\dagger)^2I$ for all $x\ge 0$;
(3^0) Q_x and L_x decrease strongly to zero as x increases to infinity.
(4^o) Suppose further that $C(iyI+A)^{-1}$ is a Hilbert–Schmidt operator for all $y\in\mathbf{R}$ and that $\int_{-\infty}^\infty\|C(iyI+A)^{-1}\|^2_{c^2}dy<\infty$. Then Q_x is trace class for each $x>0$, and

$$\operatorname{trace}Q_0=\frac{1}{2\pi}\int_{-\infty}^\infty\|C(ixI+A)^{-1}\|^2_{c^2}\,dx. \tag{2.8}$$

(5^o) *Suppose that H_0 has finite dimension m. Then* $\mathrm{rank}\frac{\partial Q_x}{\partial x} \leq m$.

Proof. (2^0),(3^0) The integrals converge by the definition of admissibility, and the other statements are immediate consequences.

(1^o) For $\xi \in D(A^\dagger)$, the $e^{-sA^\dagger}\xi$ is differentiable in H with derivative $-e^{-sA^\dagger}A^\dagger\xi$. By the fundamental theorem of calculus, we have

$$-AL_x - L_xA^\dagger = \int_x^\infty \frac{d}{ds}\big(e^{-sA}BB^\dagger e^{-sA^\dagger}\big)\,ds = -e^{-xA}BB^\dagger e^{-xA^\dagger}, \tag{2.9}$$

as bilinear forms on $D(A^\dagger) \times D(A^\dagger)$, hence the result.

(4^0) Let $(e_j)_{j=1}^\infty$ be an orthonormal basis for H. By Plancherel's formula in Hilbert space, we have

$$\int_0^\infty \|Ce^{-yA}e_j\|^2_{H_0}dy = \frac{1}{2\pi}\int_{-\infty}^\infty \|C(ixI+A)^{-1}e_j\|^2_{H_0}\,dx \tag{2.10}$$

and summing this identity we deduce

$$\sum_{j=1}^\infty \langle Q_0e_j, e_j\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty \sum_{j=1}^\infty \|C(ixI+A)^{-1}e_j\|^2_{H_0}\,dx \tag{2.11}$$

and hence

$$\mathrm{trace}\,Q_0 = \|Q_0^{1/2}\|^2_{c^2} = \frac{1}{2\pi}\int_{-\infty}^\infty \|C(ixI+A)^{-1}\|^2_{c^2}\,dx, \tag{2.12}$$

so Q_0 is trace class.

(5^0) From the expression

$$\frac{\partial Q_x}{\partial x} = -e^{-xA^\dagger}C^\dagger Ce^{-xA} \tag{2.13}$$

it follows that the rank of $\frac{\partial Q_x}{\partial x}$ is less than or equal to the rank of C, hence is less than or equal to m.

$\square$

Proposition 2.2 (Determinant of the observability Gramian).
Suppose that (4^0) holds, so that the observability operator $\Theta : L^2((0,\infty); H_0) \to H$ is Hilbert–Schmidt, where

$$\Theta f = \int_0^\infty e^{-sA^\dagger}C^\dagger f(s)\,ds \qquad (f \in L^2((0,\infty); H_0)). \tag{2.14}$$

(i) Then

$$\det(I - \lambda Q_x) = \det(I - \lambda P_{(x,\infty)}\Theta^\dagger\Theta P_{(x,\infty)}) \qquad (\lambda \in \mathbf{C}, x \geq 0). \tag{2.15}$$

defines an entire function that has all its zeros on the positive real axis.

(ii) Suppose further that $A = A^\dagger$. Then the Hankel operator Γ_ϕ on $L^2((0,\infty); H_0)$ with kernel $\phi(s+t) = Ce^{-(s+t)A}C^\dagger$, has $\Gamma_\phi = \Theta^\dagger\Theta \geq 0$ and

$$\frac{\partial}{\partial x}\mathrm{trace}\, Q_x = -\mathrm{trace}\, \phi(2x).$$

(iii) Suppose still further that $H_0 = \mathbf{C}^m$ where $m < \infty$. Then the zeros of $\det(I - \lambda Q_x)$ have order less than or equal to m.

Proof. (i) We have $\Theta^\dagger \xi(t) = Ce^{-tA}\xi$ and hence $\Theta\Theta^\dagger = Q_0$. Further, since the operators Θ and $\Theta^\dagger$ are Hilbert–Schmidt, we can rearrange terms in the determinant and obtain

$$\det(I - \lambda P_{(x,\infty)}\Theta^\dagger\Theta P_{(x,\infty)}) = \det(I - \lambda\Theta P_{(x,\infty)}\Theta^\dagger) = \det(I - \lambda Q_x). \tag{2.16}$$

The zeros of $\det(I - \lambda Q_x)$ are $1/\lambda_j$, where λ_j are the positive eigenvalues of Q_x.

(ii) Now

$$\Theta^\dagger\Theta f(t) = Ce^{-tA}\int_0^\infty e^{-sA^\dagger}C^\dagger f(s)\, ds, \tag{2.17}$$

so $\Theta^\dagger\Theta$ reduces to a Hankel operator when $A = A^\dagger$. Further, we have

$$\begin{aligned}\mathrm{trace}\, Q_x &= \int_x^\infty \mathrm{trace}\, e^{-tA}C^\dagger Ce^{-tA}\, dt\\ &= \int_x^\infty \mathrm{trace}\, Ce^{-2tA}C^\dagger\, dt,\end{aligned} \tag{2.18}$$

whence the result.

(iii) The (block) Hankel operator with kernel $\phi_{(x)}(s+t) = Ce^{(s+t+2x)A}C^\dagger$ is non negative and compact, and is unitarily equivalent to the some matrix $[a_{j+k}]_{j,k=1}^\infty$ which is made up of $m \times m$ blocks. Hence its spectrum consists of 0 together with a sequence of eigenvalues λ_j of multiplicity less than or equal to m which decrease strictly to 0 as $j \to \infty$ by [14, Theorem 2]. Hence the zeros of the function $\det(I - \lambda P_{(x,\infty)}\Theta^\dagger\Theta P_{(x,\infty)})$ have order less than or equal to m at the points $1/\lambda_j$.

□

To express the hypotheses of Proposition 2.2(ii) in terms of spectra, we present the following result, which is known to specialists.

Definition *(Carleson measure).* Let μ be a positive Radon measure on $\mathbf{C}_+ = \{z \in \mathbf{C} : \Re z > 0\}$. Then μ is a Carleson measure if there exists $c_0 > 0$ such that

$$\mu\big([0,x] \times [y-x, y+x]\big) \leq c_0 x \qquad (x > 0, y \in \mathbf{R}). \tag{2.19}$$

Proposition 2.3. *Suppose that A is self-adjoint and has purely discrete spectrum (κ_j), with $\kappa_j > 0$ listed according to multiplicity, and that (e_j) is corresponding orthonormal basis of eigenvectors. Let $\phi(x) = Ce^{-xA}C^\dagger$.*
(i) Then Γ_ϕ is bounded if and only if $\sum_{j=1}^\infty |Ce_j|^2\delta_{\kappa_j}$ is a Carleson measure.
(ii) If $\sum_{j=1}^\infty |Ce_j|^2/\kappa_j$ converges, then Γ_ϕ is of trace class.

Proof. (i) We use hats to denote Laplace transforms, and let H^2 be the usual Hardy space on $\mathbf{C}_+$ as in [10]. By the Paley–Wiener theorem, the Laplace transform gives a unitary map from $L^2(0,\infty)$ onto H^2. Then

$$\begin{aligned}\langle \Gamma_\phi f, f\rangle &= \int_0^\infty \int_0^\infty \sum_{j=1}^\infty |Ce_j|^2 e^{-(s+t)\kappa_j} f(s)\overline{f(t)}dsdt \\ &= \sum_{j=1}^\infty |Ce_j|^2 |\hat f(\kappa_j)|^2. \end{aligned} \tag{2.20}$$

Hence Γ_ϕ is bounded if and only if there exists c_1 such that

$$\Big\langle \sum_{j=1}^\infty |Ce_j|^2 \delta_{\kappa_j}, |\hat f|^2\Big\rangle \leq c_1 \lim_{x\to 0+} \int_{-\infty}^\infty |\hat f(x+iy)|^2 dy; \tag{2.21}$$

which holds if and only if we have a Carleson measure; see [10].

(ii) Note that $\sqrt{2\kappa_j}/(z+\kappa_j)$ is a unit vector in H^2, so $\hat f \mapsto 2\kappa_j \hat f(\kappa_j)/(z+\kappa_j)$ has rank one and norm one as an operator on H^2; hence the result by convexity.

□

Definition *(Balanced system).* If $Q_0 = K_0$, then the system is balanced.

Remarks

(i) The controllability operator $\Xi : L^2((0,\infty); H_0) \to H$

$$\Xi f = \int_0^\infty e^{-tA} B f(t)\, dt \tag{2.22}$$

satisfies an obvious analogue of Proposition 2.2. Note that $\Gamma_\phi = \Theta^\dagger \Xi$.

(ii) One can interchange the controllability and observability operators by interchanging $(-A, B, C) \leftrightarrow (-A^\dagger, C^\dagger, B^\dagger)$, which interchanges $\Gamma_\phi \leftrightarrow \Gamma_\phi^\dagger$.

However, we will consider in section 5 some self-adjoint Hankel operators which do not arise from the special case of $A = A^\dagger$ and $B = C^\dagger$.

(iii) The set $\mathbf{K}$ of kernels that satisfy $(2^o), (3^0), (4^o)$ and (5^o) is convex; further, for $K \in \mathbf{K}$ and $U \in B(H)$ such that $\|U\| \leq 1$, we have $U^\dagger K U \in \mathbf{K}$.

(iv) If (1^o) holds with a finite-dimensional H, then (5^o) holds. But (5^o) is not implied by finite-dimensionality of H_0.

(v) For $x > 0$, the shifted system $(-A, e^{-xA}B, Ce^{-xA})$ has observability operator Q_x, controllability operator L_x and, with $\phi_{(x)}(t) = Ce^{-(2x+t)A}B$, the corresponding Hankel operator is $\Gamma_{\phi_{(x)}}$.

Proposition 2.4. (Determinants involving the Hankel operator) *Suppose that the controllability operator Θ_x and the observability operator Ξ_x for $(-A, e^{-xA}B, e^{-xA}C)$ are Hilbert–Schmidt. Then the operator $R_x : H \to H$, defined by*

$$R_x \xi = \int_x^\infty e^{-yA} BCe^{-yA} \xi dy, \tag{2.23}$$

is of trace class and satisfies

$$\det(I - \lambda\Gamma_{\phi_{(x)}}) = \det(I - \lambda R_x). \tag{2.24}$$

Proof. By Proposition 2.2, the operator R_x is trace class. By rearranging, we obtain

$$\begin{aligned}\det(I - \lambda\Gamma_{\phi_{(x)}}) &= \det(I - \lambda\Theta_x^\dagger \Xi_x)\\ &= \det(I - \lambda \Xi_x \Theta_x^\dagger)\\ &= \det(I - \lambda R_x).\end{aligned} \tag{2.25}$$

□

In section 3 we introduce some kernels that arise from Hankel operators as in Proposition 2.2. The kernels are defined with symmetric Hamiltonian systems, as we recall in section 3.

3. Kernels arising from Hamiltonian systems of ordinary differential equations

Let D be a domain that is symmetrical with respect to the real axis and contains $(0,\infty)$. We later define kernels K_λ that satisfy some of the following properties:
(6^o) K_λ defines a bounded linear operator on H for all $\lambda \in D$;
(7^o) $\lambda \mapsto K_\lambda$ is analytic on D;
(8^o) $K_{\bar\lambda} = K_\lambda^\dagger$ for all $\lambda \in D$;
(9^o) K_λ is a Hilbert–Schmidt operator for all $\lambda \in D$;
(10^o) K_λ is an integrable operator on $L^2(I; dx)$, for some interval I; so there exist locally bounded and measurable functions ψ_k and ξ_k such that

$$K_\lambda(x,y) = \sum_{j=1}^m \frac{\psi_j(x;\lambda)\xi_j(y;\lambda)}{x-y} \qquad (x, y \in I; x \neq y)$$

and $\sum_{j=1}^m \psi_j(x;\lambda)\xi_j(x;\lambda) = 0$.

Definition *(Hamiltonian system).* Let $E(x)$ and $F(x)$ be $(2m)\times(2m)$ real symmetric matrices for each $x > 0$ such that $x \mapsto E(x)$ and $x \mapsto F(x)$ are continuous. Then we consider the symmetric Hamiltonian system

$$J\frac{d}{dx}\Psi_\lambda(x) = \big(\lambda E(x) + F(x)\big)\Psi_\lambda(x) \tag{3.1}$$

where $\Psi_\lambda(x)$ is a $(2m)\times 1$ complex vector. Suppose that for some $\lambda \in \mathbf{C}$, the solution Ψ_λ of (1.4) belongs to $L^\infty((0,\infty); \mathbf{C}^m)$. With the bilinear form $\langle (z_j), (w_j)\rangle = \sum_{j=1}^{2m} z_j w_j$, let

$$K_\lambda(x,y) = \frac{\Psi_\lambda^T(y) J \Psi_\lambda(x)}{x-y} = \frac{\langle J\Psi_\lambda(x), \Psi_\lambda(y)\rangle}{x-y}; \tag{3.2}$$

as in l'Hôpital's rule, the diagonal of the kernel is taken to be

$$K_\lambda(x,x) = \langle \Psi_\lambda(x), (\lambda E(x) + F(x))\Psi_\lambda(x)\rangle.$$

Proposition 3.1. *(i) Suppose that the solution Ψ_λ belongs to $L^\infty((0,\infty);\mathbf{C}^{2m})$ for all $\lambda\in D$. Then K_λ defines the kernel of a bounded linear operator K_λ on $L^2((0,\infty);\mathbf{C})$ such that $K_\lambda^\dagger = K_{\bar\lambda}$, so $(6^o),(7^o),(8^o)$ and (10^o) hold.*
(ii) Suppose that E and F are bounded and that Ψ_λ is a solution in $L^2((0,\infty);\mathbf{C}^{2m})$ of (3.2). Then $K_\lambda(x,y)$ defines a Hilbert–Schmidt kernel on $L^2((0,\infty);dx)$, so (9^o) also holds.

Proof. (i) Indeed, the Hilbert transform H with kernel $1/(\pi(x-y))$ is bounded on $L^2(\mathbf{R})$ and K_λ is a composition of H and bounded multiplication operators. The conditions $(6^o),(7^o)$ and (8^o) follow from basic facts about differential equations as in [8].

Observe that $\langle J\Psi_\lambda(x),\Psi_\lambda(x)\rangle=0$, since we have the bilinear product; so the formula for K_λ extends by continuity to a continuous function on $(0,\infty)^2$ and K_λ is an integrable kernel as in (10^o).

(ii) There exist constants c_1 and c_2 such that $\Vert E(x)\Vert\leq c_1$ and $\Vert F(x)\Vert\leq c_2$; hence the differential equation gives $\Vert \Psi_\lambda'(x)\Vert\leq (c_1\vert\lambda\vert+c_2)\Vert\Psi_\lambda(x)\Vert$. We deduce that Ψ_λ' belongs to $L^2((0,\infty);\mathbf{C}^{2m})$, and it is then easy to see that Ψ_λ is bounded. A further application of the differential equation shows that Ψ_λ' is also bounded.

We split the Hilbert–Schmidt integral as

$$\int_0^\infty\int_0^\infty \vert K_\lambda(x,y)\vert^2 dxdy \leq \iint_{\vert x-y\vert\geq 1}\frac{\Vert\Psi_\lambda(x)\Vert^2\Vert\Psi_\lambda(y)\Vert^2}{\vert x-y\vert^2}dxdy$$

$$+\iint_{\vert x-y\vert\leq 1}\Big\vert\Big\langle J\Psi_\lambda(x),\frac{\Psi_\lambda(x)-\Psi_\lambda(y)}{x-y}\Big\rangle\Big\vert^2 dxdy. \tag{3.3}$$

The preceding estimates show that both of these integrals converge.

□

Definition *(Shift).* For $t>0$, let $S_t: L^2(0,\infty)\to L^2(0,\infty)$ be the shift $S_tf(x)=f(x-t)$, so that $S_t^\dagger S_t=I$, and $S_tS_t^\dagger=P_{(t,\infty)}$ is the orthogonal projection onto $L^2[t,\infty)\subset L^2[0,\infty)$. In the remainder of this section, we are concerned with the effect of the shift on solutions $S_t:\Psi_\lambda(x)\mapsto\Psi_\lambda(x-t)$, and consequently on the kernels $K_{t,\lambda}=S_t^\dagger K_\lambda S_t$.

Definition. For I an interval in $\mathbf{R}$, let $L: I\times I\to M_m(\mathbf{C})$ be a continuous function. We write $L\succeq 0$ if there exist continuous functions $E_j: I\to M_m(\mathbf{C})$ for $j=1,2,\dots$ such that $\sup_{x\in I}\Vert\sum_{j=1}^\infty E_j(x)^\dagger E_j(x)\Vert<\infty$ and

$$L(x,y)=\sum_{j=1}^\infty E_j(y)^\dagger E_j(x)\qquad (x,y\in I). \tag{3.4}$$

Lemma 3.2. *Let ν be probability measure ν on I, and suppose that $L\succeq 0$ on $I\times I$.*
(i) Then $\Phi: L^2(\nu;\mathbf{C}^m)\times L^2(\nu;\mathbf{C}^m)\to\mathbf{C}$, is a positive sesquilinear form, where

$$\Phi(\xi,\eta)=\iint_{I\times I}\langle L(x,y)\xi(x),\eta(y)\rangle_{\mathbf{C}^m}\nu(dx)\nu(dy)\qquad (\xi,\eta\in L^2(I;\nu;\mathbf{C}^m)).$$

(ii) Suppose further that the defining sum (3.4) for L is finite. Then $\Phi: L^2(\nu;\mathbf{C})\to L^2(\nu;\mathbf{C}^m)$ has finite rank.

Proof. (i) The kernel is positive since

$$\sum_{k,\ell} a_{k,\ell}\Phi(\xi_k,\xi_\ell)=\sum_{j=1}^{\infty}\sum_{k,\ell} a_{k,\ell}\Big\langle \int_I E_j(x)\xi_k(x)\nu(dx),\int_I E_j(y)\xi_\ell(y)\,\nu(dy)\Big\rangle \geq 0.$$

(ii) This is clear, since Φ can be expressed a finite tensor.

□

Proposition 3.3. *Suppose that for some $\lambda>0$ there exists a bounded and continuous solution $\Psi_\lambda\in L^2((0,\infty);\mathbf{R}^{2m})$ to (3.2), where the coefficients $E(x)$ and $F(x)$ are bounded and satisfy*

$$\frac{E(x)-E(y)}{x-y}\succeq 0,\qquad \frac{F(x)-F(y)}{x-y}\succeq 0\qquad (x,y\in(0,\infty)). \tag{3.5}$$

(i) Then $K_{t,\lambda}=S_t^\dagger K_\lambda S_t$ satisfies $(1^o),(2^o),(3^o),(6^o)$,$(7^o),(8^0),(9^o)$ and (10^o).
(ii) The kernel $K_{t,\lambda}$ is of trace class, as in (4^o) and satisfies

$$\mathrm{trace}K_{t,\lambda}=\int_t^\infty\langle\Psi_\lambda(x),(\lambda E(x)+F(x))\Psi_\lambda(x)\rangle\,dx.$$

(iii) If the sums involved in (3.4) for $(E(x)-E(y))/(x-y)$ and $(F(x)-F(y))/(x-y))$ are finite, then (5^0) also holds.

Proof. (i) We observe that if $L\succeq 0$, then $\langle L(x,y)\xi,\xi\rangle$ gives the kernel of a positive definite operator. $K_\lambda(z+t,w+t)$ gives the kernel that represents $S_t^\dagger K_\lambda S_t$, and hence satisfies the Lyapunov equation

$$\frac{\partial}{\partial t}K_{t,\lambda}=-AK_{t,\lambda}-K_{t,\lambda}A^\dagger$$

where $-A=\frac{\partial}{\partial x}$ generates the semigroup $S_t^\dagger$.

From the differential equation, we have

$$\begin{aligned}\frac{\partial}{\partial t}K_\lambda(x+t,y+t)=&-\lambda\Big\langle\frac{E(x+t)-E(y+t)}{x-y}\Psi_\lambda(x+t),\Psi_\lambda(y+t)\Big\rangle\\ &-\Big\langle\frac{F(x+t)-F(y+t)}{x-y}\Psi_\lambda(x+t),\Psi_\lambda(y+t)\Big\rangle,\end{aligned} \tag{3.6}$$

and by the hypotheses on E and F we deduce that there exist $E_j,F_j:(0,\infty)\to M_{2m}(\mathbf{C})$ such that

$$\begin{aligned}\frac{\partial}{\partial t}K_\lambda(x+t,y+t)=&-\lambda\sum_{j=1}^\infty\langle E_j(x+t)\Psi_\lambda(x+t),E_j(y+t)\Psi_\lambda(y+t)\rangle\\ &-\sum_{j=1}^\infty\langle F_j(x+t)\Psi_\lambda(x+t),F_j(y+t)\Psi_\lambda(y+t)\rangle.\end{aligned} \tag{3.7}$$

The right-hand side gives the kernel of a negative definite operator, so

$$\frac{\partial}{\partial t}\langle S_t^\dagger K_\lambda S_t f, f\rangle \leq 0 \tag{3.8}$$

for all $f\in L^2(0,\infty)$.

By arguing as in Proposition 3.2, we see that $K_\lambda : L^2(0,\infty)\to L^2(0,\infty)$ is compact. For $f\in L^2(0,\infty)$, we observe that $S_tf\to 0$ weakly as $t\to\infty$ and since K_λ is compact $K_\lambda S_t f\to 0$ in norm as $t\to\infty$; hence $\langle S_t^\dagger K_\lambda S_t f, f\rangle\to 0$. Consequently $\langle K_{t,\lambda}f,f\rangle$ decreases to 0 as $t\to\infty$. Further, $S_t^\dagger K_\lambda S_t\to 0$ in Hilbert–Schmidt norm as $t\to\infty$, so there exists s_0 such that $\Vert S_t^\dagger K_\lambda S_t\Vert\leq 1$ for all $s\geq s_0$.

(ii) We have proved that $K_{t,\lambda}\geq 0$, so the kernel is positive definite and continuous. By Mercer's trace formula,

$$\mathrm{trace} K_{t,\lambda}=\int_0^\infty K_{t,\lambda}(x,x)\,dx=\int_t^\infty K_{0,\lambda}(x,x)dx. \tag{3.9}$$

(iii) If the sum over j has finitely many terms, then the expression for $\frac{\partial}{\partial t}K_\lambda(x+t,y+t)$ is a finite tensor and hence a finite-rank operator.

□

We now relate the notion of positivity from the previous definition to matrix monotonicity in Loewner's sense.

Definition *(Matrix monotone).* Let I be an open real interval and let $I^c=\mathbf{R}\setminus I$. Suppose that $E:\mathbf{C}\setminus I^c\to M_m(\mathbf{C})$ is an analytic function such that $E(x)=E(x)^\dagger$ for all $x\in I$ and

$$(E(z)-E(z)^\dagger)/(2i)=\Im E(z)\geq 0\qquad (\Im z>0). \tag{3.10}$$

Then E is a Loewner matrix function on I; equivalently, E is said to be matrix monotone.

Theorem 3.4. *Suppose that for some $\varepsilon>0$ the functions $z\mapsto E(z)$ and $z\mapsto F(z)$ are matrix Loewner functions on $(-\varepsilon,\infty)$. Suppose further that for $\Re\lambda>0$ there exists a bounded and continuous solution $\Psi_\lambda\in L^2((0,\infty);\mathbf{R}^{2m})$ to (3.1) such that $\Psi_\lambda(x)\to 0$ as $x\to\infty$.*
(i) Then for all $\Re\lambda>0$ there exists $\phi:(0,\infty)\to H_0$ and a bounded Hankel operator $\Gamma_{\phi_\lambda}:L^2(0,\infty)\to L^2((0,\infty);H_0)$ such that

$$K_\lambda=\Gamma_{\phi_\lambda}^\dagger\Gamma_{\phi_\lambda},\qquad (\Re\lambda>0) \tag{3.11}$$

and the family of kernels $K_{t,\lambda}=S_t^\dagger K_\lambda S_t$, for $\Re\lambda>0$, satisfies conditions (1^o)-(4^o), and $(6^o)-(8^o),(10^o)$.
(ii) If H_0 has finite dimension, then $K_{t,\lambda}$ also satisfies (5^o).
(iii) If $H_0=\mathbf{C}$, then Γ_{ϕ_λ} is a scalar Hankel operator and $K_\lambda=\Gamma_{\phi_\lambda}^\dagger\Gamma_{\phi_\lambda}$.
(iv) If $H_0=\mathbf{C}$ and ϕ_λ is real-valued, then $K_\lambda=\Gamma_{\phi_\lambda}^2$.

Proof. (i) We need to obtain a suitable ϕ_λ for the vectorial Hankel operator. Let $D = \mathbf{C}\setminus(-\infty,\varepsilon]$, and let $K_\lambda(z,w)$ be a kernel on $D\times D$. Then $K_\lambda(z+t,w+t)$ gives the kernel that represents $S_t^\dagger K_\lambda S_t$, and $K_\lambda = \Gamma^\dagger_{\phi_\lambda}\Gamma_{\phi_\lambda}$ if

$$\Bigl(\frac{\partial}{\partial t}\Bigr)_{t=0} K_\lambda(t+z,t+w) = \langle \phi_\lambda(z), \phi_\lambda(w)\rangle. \tag{3.12}$$

We have, from the differential equation,

$$\Bigl(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Bigr)K_\lambda(x,y) = -\lambda\Bigl\langle \frac{E(x)-E(y)}{x-y}\Psi_\lambda(x),\Psi_\lambda(y)\Bigr\rangle - \Bigl\langle \frac{F(x)-F(y)}{x-y}\Psi_\lambda(x),\Psi_\lambda(y)\Bigr\rangle. \tag{3.13}$$

By the hypotheses on E and F, there exist self-adjoint matrices $E_1, F_1 \geq 0$, self-adjoint matrices E_0 and F_0, and positive matrix measures Ω_E and Ω_F such that

$$E(x) = E_1 x + E_0 + \int_\varepsilon^\infty \Bigl(\frac{u}{1+u^2} - \frac{1}{u+x}\Bigr)\Omega_E(du), \tag{3.14}$$

and

$$F(x) = F_1 x + F_0 + \int_\varepsilon^\infty \Bigl(\frac{u}{1+u^2} - \frac{1}{u+x}\Bigr)\Omega_F(du), \tag{3.15}$$

hence (3.13) equals

$$\begin{aligned} &-\lambda\frac{E(x)-E(y)}{x-y} - \frac{F(x)-F(y)}{x-y} \\ &= -\lambda E_1 - \lambda\int_\varepsilon^\infty \frac{\Omega_E(du)}{(u+x)(u+y)} - F_1 - \int_\varepsilon^\infty \frac{\Omega_F(du)}{(u+x)(u+y)}. \end{aligned} \tag{3.16}$$

By a straightforward Hilbert space construction similar to that in [3], we can introduce H_0 and $\phi \in L^2((0,\infty);H_0)$ such that

$$\Bigl(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Bigr)\Bigl(K_\lambda(x,y)\Bigr) = -\langle \phi_\lambda(x),\phi_\lambda(y)\rangle_{H_0}. \tag{3.17}$$

Hence we have

$$\Bigl(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Bigr)\Bigl(K_\lambda(x,y) - \Gamma^\dagger_{\phi_\lambda}\Gamma_{\phi_\lambda}(x,y)\Bigr) = 0, \tag{3.18}$$

and

$$K_\lambda(x,y) - \Gamma^\dagger_{\phi_\lambda}\Gamma_{\phi_\lambda}(x,y) \to 0 \qquad (x,y\to\infty) \tag{3.19}$$

and hence

$$K_\lambda(x,y) = \Gamma^\dagger_{\phi_\lambda}\Gamma_{\phi_\lambda}(x,y) = \int_0^\infty \langle \phi_\lambda(s+x),\phi_\lambda(s+y)\rangle ds. \tag{3.20}$$

Now Γ_{ϕ_λ} is bounded since K_λ is bounded. Then for any Hankel operator $S_t^\dagger\Gamma_{\phi_\lambda} = \Gamma_{\phi_\lambda}S_t$. Hence, $S_t^\dagger\Gamma^\dagger_{\phi_\lambda}\Gamma_{\phi_\lambda}S_t = \Gamma^\dagger_{\phi_\lambda}P_{(t,\infty)}\Gamma_{\phi_\lambda}$ so that

$$K_{t,\lambda} = S_t^\dagger K_\lambda S_t = \Gamma^\dagger_{\phi_\lambda}P_{(t,\infty)}\Gamma_{\phi_\lambda} \qquad (\lambda>0). \tag{3.21}$$

(ii) When H_0 has finite dimension, the kernel $\langle \phi_\lambda(x), \phi_\lambda(y)\rangle$ has finite rank by Lemma 3.2(ii).

(iii) When $H_0 = \mathbf{C}$, the kernel of the Hankel operator is scalar-valued.

(iv) In particular, the Hankel operator with $\phi_\lambda : (0,\infty) \to \mathbf{R}$ is self-adjoint and $K_\lambda = \Gamma_{\phi_\lambda}^2$.

□

Corollary 3.5. *Suppose that $K_\lambda = \Gamma_{\phi_\lambda}^\dagger \Gamma_{\phi_\lambda}$, as in Theorem 3.4. Then under the Laplace transform $\mathcal{L} : L^2((0,\infty); \mathbf{C}^{2m}) \to H^2(\mathbf{C}_+; \mathbf{C}^{2m})$, the nullspace of K is unitarily equivalent to $\theta H^2(\mathbf{C}_+; \mathbf{C}^{2m})$ for some bounded analytic function $\theta : \mathbf{C}_+ \to M_{2m}(\mathbf{C})$ that has unitary boundary values almost everywhere.*

Proof. The null space of K_λ equals the null space of Γ_{ϕ_λ} and hence is a closed linear subspace of $L^2((0,\infty); \mathbf{C}^{2m})$ which is invariant under the shift. Beurling's theorem characterizes the images of such subspaces under the Laplace transform; see [10].

□

Asymptotic forms of the differential equation as $x, \lambda \to \infty$

We now consider circumstances under which (3.2) does have a bounded or L^2 solution Ψ_λ. Suppose that E and F are is as in Theorem 3.4 and that $F_1 = 0$ in (3.14), so that $F(x)$ is bounded. Then there are the following basic cases (i), (ii) and (iii) for the asymptotic form of (3.1) as $\lambda \to \infty$ and $x \to \infty$.

(i) Suppose that $E_1 = 0$. Then as $x \to \infty$ we have $E(x) \to \tilde{E}_0$ where

$$\tilde{E}_0 = E_0 + \int_\varepsilon^\infty \frac{u}{1+u^2} \Omega_E(du), \tag{3.22}$$

and the asymptotic form of the differential equation is

$$J \frac{d}{dx} \Phi_\lambda(x) = \lambda \tilde{E}_0 \Phi_\lambda(x) \tag{3.23}$$

with solution

$$\Phi_\lambda(x) = \exp(-\lambda x J \tilde{E}_0) \Phi_\lambda(0). \tag{3.24}$$

Now $\Re(J\tilde{E}_0) = [J, \tilde{E}_0]/2$, so $\Re(J\tilde{E}_0)$ is self-adjoint and has trace zero; hence $\Re(J\tilde{E}_0)$ is either zero, or has both positive eigenspaces and negative eigenspaces. So in the following sub-cases, the solution of (3.23) has either:

(i)(a) Φ_λ constant;

(i)(b) Φ_λ oscillating boundedly as $x \to \infty$; or

(i)(c) exponentially decaying solutions and exponentially growing solutions.

In sub-cases (i)(b) and (i)(c) there exist bounded solutions Φ_λ to (3.23) such that

$$K_\lambda(x,y) = \frac{\langle J\Phi_\lambda(x), \Phi_\lambda(y)\rangle}{x-y} \tag{3.25}$$

gives a bounded linear operator on $L^2(0,\infty)$.

(ii) Suppose that E_1 is strictly positive definite. Then the asymptotic form of the differential equation is

$$J\frac{d}{dx}\Phi_\lambda(x) = \lambda x E_1 \Phi_\lambda(x) \tag{3.26}$$

with solution

$$\Phi_\lambda(x) = \exp(-\lambda x^2 E_1/2)\Phi_\lambda(0), \tag{3.27}$$

and we have sub-cases (a), (b) and (c) analogous to those in (i) above.

(iii) E_1 of rank $1, \dots, 2m-1$. This case includes variants of Airy's equation.

Examples 3.6. (i) Sonine considered the one-parameter families of functions Y_ν that satisfy the system

$$\begin{aligned} Y_{\nu-1} + Y_{\nu+1} &= \frac{2\nu}{z} Y_\nu, \\ Y_{\nu-1} - Y_{\nu+1} &= 2Y'_\nu, \end{aligned} \tag{3.28}$$

as in [21, p 82]; the Bessel functions $Y_\nu = \mathrm{J}_\nu$ give solutions. One can transform the differential equation for J_ν into the system

$$\frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = J \begin{bmatrix} -1/x - (1-\nu^2)/4x^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{3.29}$$

which has solution $u(x) = \sqrt{x}\mathrm{J}_\nu(2\sqrt{x})$. This system is matrix monotone when $\nu = 1$.

(ii) Theorem 3.4 applies to the Airy kernel (1.3), which describes the *soft edge* of certain matrix ensembles. Likewise, the Bessel kernel describes the hard edge; see [3, 4] for details.

4. Determinantal random point fields

In section 3 we showed how some important kernels factorize as $K = \Gamma^\dagger\Gamma$. Here we consider the properties of $\det(I - \lambda\Gamma^\dagger\Gamma)$; so first we introduce and solve the Gelfand–Levitan integral equation.

Lemma 4.1. *Suppose that $-A : D(A) \subseteq H \to H$ is a generator of a bounded C_0 semigroup, $B : H_0 \to D(A)$, and $C : D(A) \to H_0$ are linear operators, where H_0 has finite dimension m, and let $\phi(x) = Ce^{-xA}B$. Suppose further that $K_C(A), K_{B^\dagger}(A^\dagger) \leq 1$.*
(i) Then the $m \times m$ matrix kernel

$$T_\lambda(x,y) = -\lambda C e^{-xA}\big(I + \lambda R_x\big)^{-1} e^{-yA} B \qquad (0 < x < y, |\lambda| < 1) \tag{4.1}$$

gives the unique solution of the integral equation

$$T_\lambda(x,y) + \phi(x+y) + \lambda \int_x^\infty T_\lambda(x,z)\phi(z+y)\,dz = 0 \qquad (0 < x < y), \tag{4.2}$$

and the kernel $T_\lambda(x,y)$ satisfies

$$\frac{\partial^2 T_\lambda}{\partial x^2} - \frac{\partial^2 T_\lambda}{\partial y^2} - q(x)T_\lambda(x,y) = 0 \tag{4.3}$$

where $q(x) = -2\frac{d}{dx}T_\lambda(x,x)$.
(ii) Suppose further that $m = 1$, *and that* Θ_x *and* Ξ_x *are Hilbert–Schmidt. Then the determinant satisfies*

$$T_\lambda(x,x) = \frac{d}{dx}\log\det(I + \lambda\Gamma_{\phi_{(x)}}) \qquad (x > 0). \tag{4.4}$$

Proof. (i) First, we have $\|R_x\| = \|\Xi_x\Theta_x^\dagger\| \leq 1$, so $I + \lambda R_x$ is invertible and $T_\lambda(x,y)$ is well defined. One checks the identity by substituting the given expression for T_λ into the integral equation. Further, $\|\Gamma_{\phi_{(x)}}\| = \|\Theta_x^\dagger\Xi_x\|$, so $I - \lambda\Gamma_{\phi^\dagger_{(x)}}$ is invertible so hence solutions to the Gelfand–Levitan integral equation are unique.

One can differentiate the integral equation and integrate by parts to obtain

$$\frac{\partial^2 T_\lambda}{\partial x^2} - \frac{\partial^2 T_\lambda}{\partial y^2} + \lambda q(x)\phi(x+y) + \lambda\int_x^\infty \Bigl(\frac{\partial^2 T_\lambda}{\partial x^2} - \frac{\partial^2 T_\lambda}{\partial z^2}\Bigr)\phi(z+y)\,dz = 0, \tag{4.5}$$

so by uniqueness

$$\frac{\partial^2 T_\lambda}{\partial x^2} - \frac{\partial^2 T_\lambda}{\partial y^2} = q(x)T_\lambda(x,y). \tag{4.6}$$

(ii) When $H_0 = \mathbf{C}$ the kernel takes values in $\mathbf{C}$. Here $R_x = \Xi_x\Theta_x^\dagger$ is trace class, and we can rearrange the traces and compute

$$\begin{aligned} T_\lambda(x,x) &= -\lambda\mathrm{trace}\bigl(Ce^{-xA}(I+\lambda R_x)^{-1}e^{-xA}B\bigr)\\ &= -\lambda\mathrm{trace}\bigl((I+\lambda R_x)^{-1}e^{-xA}BCe^{-xA}\bigr)\\ &= \frac{d}{dx}\mathrm{trace}\log(I+\lambda R_x)\\ &= \frac{d}{dx}\log\det(I+\lambda\Gamma_{\phi_{(x)}}), \end{aligned} \tag{4.7}$$

where the last step follows from Proposition 2.2.

□

Our first application is to the context of Theorem 1.2(i), where we consider determinantal random point fields associated with the observability Gramian.

Theorem 4.2. *Suppose that* (4^o) $\Theta : L^2((0,\infty);\mathbf{C}) \to H$ *defines a Hilbert–Schmidt operator, and that* (2^o) *the operator norm is* $\|\Theta\| < 1$.
(i) Then there exists a determinantal random point field on $(0,\infty)$ *such that* $\nu(x,\infty)$ *is the number of points in* (x,∞) *and such that the generating function satisfies*

$$\mathbf{E}z^{\nu(x,\infty)} = \det(I + (z-1)Q_x) \qquad (z \in \mathbf{C}, x > 0). \tag{4.8}$$

(ii) Let F *be the cumulative distribution function*

$$F(x) = \begin{cases} \mathbf{P}[\nu(x,\infty) = 0] & (x \geq 0)\\ 0 \qquad (x < 0). \end{cases} \tag{4.9}$$

Then

$$F'(x)/F(x) = \mathrm{trace}((A + A^\dagger)Q_x(I - Q_x)^{-1}) \qquad (x > 0). \tag{4.10}$$

(iii) In particular, if $A = A^\dagger$, then $\det(I + (z-1)\Gamma_{\phi_{(x)}})$ gives a generating function.
(iv) When $A = A^\dagger$, the kernels

$$T_\lambda(x, y) = -\lambda C e^{-xA}(I + \lambda Q_x)^{-1} e^{-yA} C^\dagger \qquad (0 < x < y, |\lambda| < 1) \tag{4.11}$$

and $\phi(x + y) = Ce^{-(x+y)A}C^\dagger$ satisfy the Gelfand–Levitan integral equation

$$T_\lambda(x, y) + \lambda\phi(x + y) + \lambda \int_x^\infty T_\lambda(x, z)\phi(z + y)\, dz = 0 \tag{4.12}$$

and the diagonal satisfies

$$T_\lambda(x, x) = \frac{d}{dx} \log\det(I + \lambda\Gamma_{\phi_{(x)}}). \tag{4.13}$$

Proof. (i) The kernel $Ce^{-sA}e^{-tA^\dagger}C^\dagger$ of $\Theta^\dagger\Theta$ gives an integral operator on $L^2(0, \infty)$ such that $0 \leq \Theta^\dagger\Theta \leq I$; hence by Lemma 1.1 is associated with a determinantal random point field such that

$$\begin{aligned} \mathbf{E}z^{\nu(x,\infty)} &= \det(I + (z-1)\Theta^\dagger\Theta P_{(x,\infty)}) \\ &= \det(I + (z-1)\Theta P_{(x,\infty)}\Theta^\dagger) \\ &= \det(I + (z-1)Q_x), \end{aligned} \tag{4.14}$$

so the determinant involving the observability Gramian gives rise to the determinantal random point field.

(ii) We consider the probability that all of the random points lie in $(0, x)$. The operator $I - Q_x$ is invertible since $\|Q_x\|_{op} < 1$, and we have

$$F(x) = \det(I - Q_x) \qquad (x > 0). \tag{4.15}$$

By a familiar formula for determinants, we have

$$\begin{aligned} \frac{d}{dx}\log\det(I - Q_x) &= \frac{d}{dx}\mathrm{trace}\log(I - Q_x) \\ &= -\mathrm{trace}\Big((I - Q_x)^{-1}\frac{d}{dx}Q_x\Big) \\ &= \mathrm{trace}\Big((I - Q_x)^{-1}(A^\dagger Q_x + Q_x A)\Big) \geq 0, \end{aligned} \tag{4.16}$$

where the last step follows from the Lyapunov equation (1^o). Condition (3^o) reassures us that $F(x)$ is indeed an increasing function, and that the associated probability density function satisfies (4.10).

(iii) If $A = A^\dagger$, then $Ce^{-(s+t)A}C^\dagger$ is the kernel of $\Gamma_{\phi_{(x)}} = \Theta_x^\dagger \Theta_x$, so

$$\det(I + (z-1)Q_x) = \det(I + (z-1)\Gamma_{\phi_{(x)}}). \tag{4.17}$$

(iv) This is a special case of Lemma 4.1.

□

Now we state the variant which arises in random matrix theory as in Theorem 3.4(iv) and Theorem 1.2(ii).

Theorem 4.3. *Suppose that A, B and C satisfy the hypotheses of Lemma 4.1, and that $\phi = \phi^\dagger$. Let $\phi_{(x)}(y) = \phi(2x + y)$.*
(i) Then there exists a determinantal random point field on $(0, \infty)$ such that $\nu(x, \infty)$ is the number of points in (x, ∞) such that the generating function satisfies

$$\mathbf{E} z^{\nu(x,\infty)} = \det(I + (z-1)\Gamma^2_{\phi_{(x)}}) \qquad (z \in \mathbf{C}, x > 0). \tag{4.18}$$

(ii) Further,

$$\frac{d}{dx} \log\det(I - \lambda^2 \Gamma^2_{\phi_{(x)}}) = T_{-\lambda}(x,x) + T_\lambda(x,x) \qquad (|\lambda| < 1), \tag{4.19}$$

where $T_\lambda(x,y) = -\lambda C e^{-xA}(I + \lambda R_x)^{-1} e^{-yA} B$ satisfies a Gelfand–Levitan equation as in (4.2).

Proof. First we check that $K_x = \Gamma^2_{\phi_{(x)}}$ satisfies $(2^o), (4^0)$ and (5^0). We have $\|\Gamma_{\phi_{(x)}}\| \leq 1$, so $0 \leq K_x \leq I$, and $K_x = \Theta_x^\dagger R_x$ is of trace class. Hence by Soshnikov's theorem, we can form a determinantal random point field with generating function as above.

To calculate the determinant, one can use the identity

$$\begin{aligned} \log\det(I - \lambda^2 K_x) &= \log\det(I - \lambda\Gamma_{\phi_{(x)}}) + \log\det(I + \lambda\Gamma_{\phi_{(x)}}), \\ &= \log\det(I - \lambda R_x) + \log\det(I + \lambda R_x). \end{aligned} \tag{4.20}$$

(ii) The terms on the right-hand side satisfy

$$\frac{d}{dx}\big(\log\det(I - \lambda R_x) + \log\det(I + \lambda R_x)\big) = T_{-\lambda}(x,x) + T_\lambda(x,x).$$

by the Gelfand–Levitan equation. Indeed we can replace B in Lemma 4.1 by $\pm B$ to introduce $\pm\lambda\Gamma_{\phi_{(x)}}$.

□

We defer discussion of Theorem 1.2(iii) until section 6. In section 5, we consider the determinant in Theorem 4.3 from the perspective of scattering theory.

5. Scattering and inverse scattering

The Gelfand–Levitan integral equation of Lemma 4.1 is closely connected to the Schrödinger equation, as we discuss in this section. Our aim is to identify a group of

bounded linear operators which acts naturally on the ϕ that appear in Theorem 4.3, and hence on the determinants.

Given $\phi(x) = Ce^{-xA}B$ as in Lemma 4.1, we can solve the Gelfand–Levitan equation and recover $q(x) = -2\frac{d}{dx}T_\lambda(x,x)$. Further, given T_λ as in Lemma 4.1, the function

$$\psi(x;k) = e^{ikx} + \int_x^\infty e^{iyk}T(x,y)dy \tag{5.1}$$

satisfies

$$-\frac{d^2}{dx^2}\psi(x;k) + q(x)\psi(x;k) = k^2\psi(x;k) \tag{5.2}$$

and

$$\psi(x;k) \asymp e^{ikx} \qquad (x \to \infty). \tag{5.3}$$

This is a straightforward calculation, based upon (2.25).

(i) Let $(\lambda_j)_{j=1}^n$ be the discrete spectrum of $-\frac{d^2}{dx^2} + q$ in $L^2(\mathbf{R})$, written $\lambda_j = -\kappa_j^2$ with $\kappa_j > 0$ so that each $\lambda_j = -\kappa_j^2$ is associated with an eigenfunction $\psi(x;\lambda_j)$ that is asymptotic to $e^{-\kappa_j x}$ as $x \to \infty$ and $\kappa_n \geq \ldots \geq \kappa_1 > 0$. We take $c(-\kappa_j^2)$ to be a constant associated with $-\kappa_j^2$.

(ii) The continuous spectrum is $\Sigma_c = [0,\infty)$, which has multiplicity two. For $k \in \mathbf{R}$ and $\lambda = k^2 > 0$, there exists a solutions $\psi(x;k)$ to (5.2) with asymptotic behaviour

$$\psi(x,k) \asymp \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to \infty; \\ a(k)e^{-ikx} & \text{as } x \to -\infty. \end{cases} \tag{5.4}$$

By [12], the reflection coefficient b belongs to C_0^∞, satisfies $b(0) = -1$ and $b(-k) = \bar{b}(k)$ and

$$\int_{-\infty}^\infty \Big|\frac{\log(1-|b(k)|^2)}{1+k^2}\Big| dk < \infty. \tag{5.5}$$

(iii) By results from [12, 13], the transmission coefficient a extends to define the outer function

$$a(k) = \exp\Big(\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\log(1-|b(t)|^2)}{t-k}\,dt\Big) \tag{5.6}$$

on $\{k : \Im k > 0\}$ such that $|a(k)| = (1-|b(k)|^2)^{1/2}$ for $k \in \mathbf{R}$.

The scattering map $q \mapsto \phi$ associates, to the potential q, the function

$$\phi(x) = \sum_{j=1}^n c(-\kappa_j^2)^2 e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^\infty b(k)e^{ikx}\,dk. \tag{5.7}$$

where the eigenvalues $\lambda_j = -\kappa_j^2$, the normalizing constants $c(-\kappa_j^2)$ and the reflection coefficient $b(\kappa)$ are the scattering data. By (ii), $\phi(x)$ is real.

The aim of inverse scattering is to recover q, up to translation from the scattering data. The following section uses computations which are extracted from [1,2], and originate

in other calculations of inverse problems as in [6,21]. The reflection coefficient b, the negative eigenvalues λ_j and the normalising constants $c(-\kappa_j^2)$ determine q uniquely up to translation.

Our general approach to the inverse spectral problem is to go

$$\phi \mapsto (-A, B, C) \mapsto T \mapsto q. \tag{5.8}$$

We now consider the first step in the process, namely realising $(-A, B, C)$ from a given ϕ.

Definition *(Realisation).* Given an bounded linear operator Γ, we wish to find a linear system $(-A, B, C)$ such that the corresponding Hankel operator Γ_ϕ is unitarily equivalent to Γ. In particular, given scattering data $\phi(x)$, we wish to find a balanced linear system such that $\phi(x) = Ce^{-xA}B$ for $x > 0$.

The following Lemma gives a characterization up to unitary equivalence of a special class of Hankel operators.

Definition *(Spectral multiplicity).* For a self-adjoint and bounded linear operator A on H with spectrum S, let

$$H = \int_S^{\oplus} H(\lambda)\,\mu(d\lambda)$$

be the spectral resolution, where μ is a bounded positive Radon measure on S, such that $Af(\lambda) = \lambda f(\lambda)$. Now let $\delta(\lambda) = \dim H(\lambda)$ be the spectral multiplicity function for $\lambda \in S$.

Lemma 5.1. *Let Γ be a self-adjoint and bounded linear operator on H such that:*
(i) the nullspace of Γ is zero or infinite-dimensional;
(ii) Γ is not invertible;
(ii) $|\delta(\lambda) - \delta(-\lambda)| \leq 1$ for μ almost all λ.
Then there exists a balanced linear system $(-A, B, C)$ with $H_0 = \mathbf{C}$ such that the Hankel operator Γ_ϕ on $L^2(0, \infty)$ with kernel $\phi(x + y) = Ce^{-(x+y)A}B$ is unitarily equivalent to Γ.

Proof. This is a special case of Theorem 1.1 on p. 257 of [14].

□

Proposition 5.2. *Suppose that the hypotheses of Lemma 5.1 hold.*
(i) If the spectral density function b on the continuous spectrum is identically zero, then the scattering data can be realized by a linear system with finite dimensional H and H_0.
(ii) Suppose that in the corresponding linear system A is a finite matrix such that all its eigenvalues κ_j satisfy $\Re\kappa_j > 0$. Then the system is admissible.

Proof. (i) By a theorem of Fuhrmann [15], one can choose A to be a finite-rank operator if and only if the transfer function $\hat{\phi}$ is a rational function which is analytic on the closure of $\mathbf{C}_+ \cup \{\infty\}$.

(ii) The system is admissible since $\|Ce^{-tA}\xi\| \leq Me^{-\kappa t}\|\xi\|$ for some $M, \kappa > 0$ and $\xi \in H_0$.

□

Remarks. (i) Not all self-adjoint Hankel operators satisfy the condition (ii) of Lemma 5.1. Consequently, there is a distinction between those self-adjoint Hankel operators that can be realised by linear systems in continuous time with one-dimensional input and output

spaces and the more general class that can be realised by linear systems in discrete time. In this paper we concentrate on the continuous time case, while McCafferty has considered analogous results in discrete time, as in [11].

(ii) If $b = 0$ in (5.9), then Propositions 5.2 and 2.3 apply to ϕ.

Definition *(Evolution).* For a system $(-A, B, C)$, we refer to ϕ as scattering data. Given a C_0 group $E(t)$ on H and $D(A)$, we can form the system $(-A, B, CE(t))$ and introduce $\tilde{E}(t)\phi(s) = CE(t)e^{-sA}B$; thus the scattering data evolves with t.

Article [3] highlighted the importance of groups that satisfy the Weyl relations; here we show that these are associated with evolutions that do not change the determinants in Theorem 1.2.

Proposition 5.3. *Suppose that D is skew self-adjoint and generates a C_0 group of unitary operators that satisfies $e^{-sD}e^{-tA} = e^{-i\alpha st}e^{-tA}e^{-sD}$ for all $s \in \mathbf{R}, t \geq 0$ and some $\alpha \in \mathbf{R}$. Then $(-A, e^{sD}B, Ce^{-sD})$ has observability Gramian $Q^{(s)}$, operator $R^{(s)}$ and controllability Gramian $L^{(s)}$ such that (i) $\det(I + \lambda Q^{(s)})$, (ii) $\det(I + \lambda R^{(s)})$ and (iii) $\det(I + \lambda Q^{(s)}L^{(s)})$ are independent of s.*

Proof. (i) We have $e^{-tA^\dagger}e^{sD}C^\dagger Ce^{-sD}e^{-tA} = e^{sD}e^{-tA^\dagger}C^\dagger Ce^{-tA}e^{-sD}$, so $Q^{(s)} = e^{sD}Q_0e^{-sD}$ and by unitary equivalence $\det(I + \lambda Q^{(s)}) = \det(I + \lambda Q_0)$.

(ii) We have $R^{(s)} = e^{-sD}R_0e^{sD}$, so $\det(I + \lambda R^{(s)}) = \det(I + \lambda R_0)$.

(iii) Likewise, $L^{(s)} = e^{sD}L_0e^{-sD}$, and there is a unitary equivalence $Q^{(s)}L^{(s)} = e^{sD}Q_0L_0e^{-sD}$ leading to $\det(I + \lambda Q^{(s)}L^{(s)}) = \det(I + \lambda Q_0L_0)$.

□

Lemma 5.4. *Suppose that $\Psi_t(x; k)$ gives a differentiable family of vectors in $\mathbf{C}^{2m}$ such that*

$$\frac{d}{dt}\Psi_t(x; k) = Z_t(x; k)\Psi_t(x; k), \tag{5.9}$$

where

$$Z_t(x; k) = \begin{bmatrix} \alpha_t(x; k) & \beta_t(x; k) \\ -\gamma_t(x; k) & -\alpha_t(x; k) \end{bmatrix} \tag{5.10}$$

and $\alpha_t(x; k), \beta_t(x; k)$ and $\gamma_t(x; k)$ are symmetric $m \times m$ matrices.
(i) Then with the bilinear form on $\mathbf{C}^{2m}$, the family of kernels

$$K_{t,x}(\kappa, k) = \frac{\langle J\Psi_t(x; \kappa), \Psi_t(x; k)\rangle}{\kappa - k} \qquad (k, \kappa \in \mathbf{R}, k \neq \kappa)$$

satisfies (10^o) *and*

$$\frac{\partial}{\partial t}K_{t,x}(\kappa, k) = \Big\langle J\Big(\frac{Z_t(x; \kappa) - Z_t(x; k)}{\kappa - k}\Big)\Psi_t(x; \kappa), \Psi_t(x; k)\Big\rangle \tag{5.11}$$

(ii) If the $\alpha_t(x; k), \beta_t(x; k)$ and $\gamma_t(x; k)$ are rational functions of k, then $\frac{\partial}{\partial t}K_{t,x}$ is of finite rank.

Proof. (i) This follows by direct calculation, where the the effect of $\frac{\partial}{\partial t}$ is to replace J by $JZ_t(x; k) + Z_t(y; k)^TJ$ in the kernel. Then one uses the identities

$$JZ_t(x; \kappa) + Z_t(x; k)^TJ = J(Z_t(x; \kappa) - Z_t(x; k)), \tag{5.12}$$

which follow from the special form of the matrices.

(ii) Given the formula (5.21), one can use the partial fraction decomposition of the entries to express $\frac{\partial}{\partial t}K_{t,x}(\kappa, k)$ as a sum of products of functions in the variable κ or k.

$\square$

In accordance with the approach of [7], we are particularly interested in the case where $k \mapsto Z_t(x;k)$ is a polynomial such that the leading coefficient has trace zero. For Schrödinger's equation, we can introduce such families of matrices associated with the KdV flow. Let C_0^∞ be the space of functions $f : \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable and such that $|x|^j|f^{(\ell)}(x)| \to 0$ as $x \to \infty$ for $j, \ell = 0, 1, \ldots$.

Suppose that q satisfies that $q = v' + v^2$. Given a ψ that satisfies $-\psi'' + q\psi = k^2\psi$, we have a solution of the symmetric Hamiltonian system

$$\frac{d}{dx}\begin{bmatrix} \psi \\ \rho \end{bmatrix} = \begin{bmatrix} v & ik \\ ik & -v \end{bmatrix}\begin{bmatrix} \psi \\ \rho \end{bmatrix}. \tag{5.13}$$

Now let v evolve according to the modified Korteweg–de Vries equation

$$4\frac{\partial v}{\partial t} = \frac{\partial^3 v}{\partial x^3} - 6v^2\frac{\partial v}{\partial x}, \tag{5.14}$$

and introduce functions of (x, t) by

$$\alpha = (1/4)v_{xx} - (1/2)v^3, \quad \beta = (-1/2)(v_x + v^2), \quad \gamma = (1/2)(v_x - v^2), \quad \delta = v. \tag{5.15}$$

Lemma 5.5. *The matrices*

$$V_t(x;z) = \begin{bmatrix} v & z \\ z & -v \end{bmatrix} \quad \text{and} \quad Z_t(x;z) = \begin{bmatrix} \alpha + \delta z^2 & \beta z + z^3 \\ \gamma z + z^3 & -\alpha - \delta z^2 \end{bmatrix} \tag{5.16}$$

give a consistent system

$$\begin{cases} \frac{d}{dx}\Psi = V_t(x;z)\Psi, \\ \frac{d}{dt}\Psi = Z_t(x;z)\Psi. \end{cases} \tag{5.17}$$

Proof. As in [7], it follows by direct computation that

$$\frac{\partial V_t(x;z)}{\partial t} - \frac{\partial Z_t(x;z)}{\partial x} + [V_t(x;z), Z_t(x;z)] = 0, \tag{5.18}$$

so $\frac{\partial^2}{\partial x \partial t}\Psi = \frac{\partial^2}{\partial t \partial x}\Psi$ and the system is consistent. The key idea is that one can equate coefficients of the ascending powers of z, then one can eliminate the functions α, β, γ and δ by simple calculus.

$\square$

Let $\Psi_t(x;k)$ be the solution of (5.17) that corresponds to $z = ik$ where k belongs to $\mathbf{R}$ and k^2 to the continuous spectrum $(0, \infty)$. With the bilinear form $\langle . , . \rangle$ on $\mathbf{C}^2$, let

$$K_{t,x}(\kappa, k) = \frac{\langle J\Psi_t(x;\kappa), \Psi_t(x;k)\rangle}{i(\kappa - k)}. \tag{5.19}$$

where the numerator vanishes on $k=\kappa$, so $K_{t,x}$ is an integrable operator. This family of operators undergoes a natural evolution under the KdV flow, as follows.

Theorem 5.6. *Suppose that $\Psi_t(x;k)$ give a locally bounded family of solutions which is differentiable in (t,x,k) and subject to $\Psi_t(0;k)=\Psi_t$ for some $\Psi_t\in \mathbf{C}^2$.*
(i) If $v(x)=0$, then $K_{t,x}$ is a multiple of the sine kernel [3, (1.2)].
(ii) The kernels $K_{t,x}$ satisfy (10^o) and (5^o), so $\frac{\partial}{\partial x}K_{t,x}$ and $\frac{\partial}{\partial t}K_{t,x}$ are of finite rank.
(iii) The function $u=\frac{\partial v}{\partial x}+v^2$ satisfies the KdV equation, and as $q(x)$ evolves to $u(x,t)$ the scattering data for u undergoes a linear evolution $\phi\mapsto E(t)\phi$.
(iv) Let $(b,c(-\kappa_j^2),\kappa_j)$ be the scattering data for $q(x)$, and suppose that $b(k),b'(k)$ and $k^2b(k)$ belong to $L^2(\mathbf{R};dk)$. Then $\Gamma_{E(t)\phi}$ gives a Hilbert–Schmidt operator for all t.

Proof. (i) This is an elementary computation.

(ii) Using Lemma 5.4, we calculate the derivatives, and find

$$\frac{\partial K_{t,x}}{\partial t}=\Big\langle\begin{bmatrix}-\gamma+(k^2+k\kappa+\kappa^2) & i\delta(k+\kappa)\\ i\delta(k+\kappa) & \beta-(k^2+k\kappa+k^2)\end{bmatrix}\Psi_t(x;\kappa),\Psi_t(x;k)\Big\rangle, \tag{5.20}$$

which gives a kernel of finite rank, and likewise

$$\frac{\partial K_{t,x}}{\partial x}=\Big\langle\begin{bmatrix}1&0\\0&-1\end{bmatrix}\Psi_t(x;\kappa),\Psi_t(x,k)\Big\rangle \tag{5.21}$$

which also gives a kernel of finite rank on $L^2(-\infty,\infty)$.

(iii) By Miura's transformation, the function $u=\frac{\partial v}{\partial x}+v^2$ satisfies the KdV equation

$$4\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial x^3}-6u\frac{\partial u}{\partial x}; \tag{5.22}$$

see [6, p.65]. The evolution of the potentials $u(x,0)\mapsto u(x,t)$ under the KdV flow gives rise to a linear evolution on the scattering data. Now let $\Psi_t(x;k)$ be a continuous and uniformly bounded family of solutions of the system

$$\begin{cases}\frac{d}{dx}\Psi_t(x;k)=U_t(x;k)\Psi_t(x;k)\\ \frac{d}{dt}\Psi_t(x;k)=W_t(x;k)\Psi_t(x;k)\end{cases} \tag{5.23}$$

where

$$U_t(x;k)=\begin{bmatrix}0&1\\u-k^2&0\end{bmatrix},\quad W_t(x;k)=\frac{-1}{4}\begin{bmatrix}4ik^3-\frac{\partial u}{\partial x} & 2u+4k^2\\ 2(u+2k^2)(u-k^2)-\frac{\partial^2u}{\partial x^2} & 4ik^3+\frac{\partial u}{\partial x}\end{bmatrix}. \tag{5.24}$$

Then by considering the shape of the matrices in (5.24), we obtain the asymptotic forms of the solutions

$$\Psi_t(x;k)\asymp -ia(k)e^{-ikx}\begin{bmatrix}i\\k\end{bmatrix}\qquad (x\to-\infty),$$

$$\Psi_t(x;k)\asymp -ie^{-ikx}\begin{bmatrix}i\\k\end{bmatrix}+ib(k)\begin{bmatrix}-i\\k\end{bmatrix}e^{ikx-2ik^3t}\qquad (x\to\infty); \tag{5.25}$$

hence $a(k) \mapsto a(k)$ and $b(k) \mapsto b(k)e^{-2ik^3 t}$ under the flow. By [6, p. 75], there is a group of linear operators $E(t)$ on the Hilbert space $\mathbf{C}^n \oplus L^2(\mathbf{R})$ defined by

$$E(t)\phi(x) = \sum_{j=1}^{n} c(-\kappa_j^2)^2 e^{-\kappa_j x - 2\kappa_j^3 t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx-2ik^3 t}dk \tag{5.26}$$

such that $u(x,y)$ corresponds to $E(t)\phi$, and $\|E(t)\| = \max\{e^{-2t\kappa_n^3}, 1\}$.
By applying Fourier inversion to the definition of the Airy function, we can express the integral over the continuous spectrum as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx-2ik^3 t}dk = \frac{-1}{(6t)^{1/3}}\int_0^{\infty} \mathrm{Ai}\Big(\frac{x-y}{(-6t)^{1/3}}\Big)\phi(y)dy.$$

(iv) The Hankel operator with kernel $\sum_{j=1}^n c(-\kappa_j^2)^2 e^{-\kappa_j(x+y)-2\kappa_j^3 t}$ is clearly of trace class, so we need to consider the Hankel arising from b. To show that $\int_0^\infty x(E(t)\phi(x))^2 dx < \infty$, it suffices by Plancherel's theorem to show that $b(k)$ and $\frac{d}{dk}(b(k)e^{-2ik^3 t})$ belong to $L^2(\mathbf{R}; dk)$. This follows directly from the hypotheses.

□

6 Determinantal random point fields associated with the Zakharov–Shabat system

In this final section we prove the remaining case (iii) of Theorem 1.2, and then we address the corresponding scattering theory.
Consider the matricial Gelfand–Levitan integral equation

$$T(x,y) + \lambda\Phi(x+y) + \lambda\int_x^\infty T(x,z)\Phi(z+y)\,dz = 0 \qquad (0 < x < y), \tag{6.1}$$

where, suppressing the dependence of T upon λ, we write

$$T(x,y) = \begin{bmatrix} \bar U(x,y) & V(x,y) \\ -\bar V(x,y) & U(x,y) \end{bmatrix}, \qquad \Phi(x) = \begin{bmatrix} 0 & \bar\phi(x) \\ -\phi(x) & 0 \end{bmatrix}. \tag{6.2}$$

Theorem 6.1. *Suppose that the system $(-A, B, C)$ has $H_0 = \mathbf{C}$ and with $\phi_{(x)}(y) = Ce^{-(2x+y)A}B$ satisfies, as in Lemma 4.1:*
(2^o) $\|\Theta_x\| < 1$ and $\|\Xi_x\| \leq 1$, and
(4^o) Θ_x and Ξ_x are Hilbert–Schmidt.
i) Then there exists a determinantal random point field on $(0,\infty)$ such that $\nu(x,\infty)$ is the number of points in (x,∞) and such that the generating function satisfies

$$g_x(z) = \mathbf{E}z^{\nu(x,\infty)} = \det(I + (z-1)\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}}). \tag{6.3}$$

(ii) Further $\frac{\partial}{\partial x}\log g_x(z) = 2U(x,x)$, where U is given by the diagonal of the solution of the Gelfand–Levitan equation (6.1).

Proof. (i) We have $0 \leq Q_x \leq I$ and $0 \leq L_x \leq I$. Let $K_x = P_{(x,\infty)}\Theta^\dagger L_x \Theta P_{(x,\infty)}$, which defines a trace-class kernel on $L^2(x,\infty)$ and satisfies $0 \leq K_x \leq I$. Then by Lemma 1.1, there exists a determinantal random point field on (x,∞) with generating function

$$\begin{aligned}\det(I+(z-1)K_x) &= \det(I+(z-1)\Theta P_{(x,\infty)}\Theta^\dagger L_x)\\ &= \det(I+(z-1)Q_xL_x),\end{aligned}\tag{6.4}$$

and we continue to rearrange this, obtaining

$$\begin{aligned}\det(I+(z-1)K_x) &= \det(I+(z-1)P_{(x,\infty)}\Theta^\dagger \Xi P_{(x,\infty)}\Xi^\dagger\Theta)\\ &= \det(I+(z-1)P_{(x,\infty)}\Gamma_\phi P_{(x,\infty)}\Gamma_\phi^\dagger P_{(x,\infty)})\\ &= \det(I+(z-1)\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}}).\end{aligned}\tag{6.5}$$

(ii) This identity is proved in the following two Lemmas.

Lemma 6.2. *Suppose that $H_0 = \mathbf{C}$ and let $\phi(x) = Ce^{-xA}B$ and $G_x = I + \lambda^2 Q_x L_x$. Then the Gelfand–Levitan integral equation (6.1) reduces to*

$$V(x,y) + \lambda\bar\phi(x+y) + \lambda^2\int_x^\infty\int_x^\infty V(x,s)\phi(s+z)\bar\phi(y+z)dsdz = 0 \qquad (0<x<y), \tag{6.6}$$

which has solution

$$V(x,y) = -\lambda B^\dagger e^{-A^\dagger x}G_x^{-1}e^{-A^\dagger y}C^\dagger \qquad (0<x<y), \tag{6.7}$$

$$\bar U(x,y) = \lambda\int_x^\infty V(x,z)\phi(z+y)\,dz. \tag{6.8}$$

Proof. Once we have V, we can introduce U via (6.7), and the resulting matrix T satisfies the Gelfand–Levitan integral equation. To verify the equation for T, we first check that G_x is invertible when $\Re\lambda^2 > -1$. The operators Q_x and L_x are Hilbert–Schmidt and positive, so the operator Q_xL_x is trace class, and hence the determinant satisfies

$$\det G_x = \det(I+\lambda^2 Q_x^{1/2}L_xQ_x^{1/2}) > 0 \tag{6.9}$$

since $\Re(I+\lambda^2Q_x^{1/2}L_xQ_x^{1/2}) \geq (1-\Re\lambda^2)I$.

One can postulate a solution of the form $V(x,y) = X(x)^\dagger e^{-A^\dagger y}C^\dagger$, for some function $X : (0,\infty) \to H$ and by substituting this into the integral equation, one finds that X should satisfy

$$\begin{aligned}&X(x)^\dagger e^{-A^\dagger y}C^\dagger\lambda B^\dagger e^{-A^\dagger(x+y)}C^\dagger\\ &+\lambda^2\int_x^\infty\int_x^\infty X(x)^\dagger e^{-A^\dagger s}C^\dagger Ce^{-A(s+z)}BB^\dagger e^{-A^\dagger(z+y)}C^\dagger dsdz = 0,\end{aligned}\tag{6.10}$$

so we want

$$X(x)^\dagger(I+\lambda^2Q_xL_x) + \lambda B^\dagger e^{-A^\dagger x} = 0, \tag{6.11}$$

and we can make this choice since G_x is invertible.

$\square$

Lemma 6.3 *The diagonal of the solution satisfies*

$$U(x,x) = \frac{d}{dx}\frac{1}{2}\log\det(I + \lambda^2\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}}). \tag{6.12}$$

Proof. From (6.8), we have

$$\begin{aligned}\bar{U}(x,y) &= -\lambda^2\int_x^\infty B^\dagger e^{-A^\dagger x}G_x^{-1}e^{-A^\dagger z}C^\dagger C e^{-A(z+y)}B\,dz\\ &= -\lambda^2 B^\dagger e^{-A^\dagger x}G_x^{-1}Q_x e^{-Ay}B. \end{aligned}\tag{6.13}$$

Hence we can write

$$\begin{aligned}U(x,x) &= -\lambda^2 B^\dagger e^{-A^\dagger x}G_x^{-1}Q_x e^{-Ax}B\\ &= -\lambda^2\mathrm{trace}\Big(G_x^{-1}Q_x\frac{d}{dx}L_x\Big).\end{aligned}\tag{6.14}$$

We temporarily assume that λ is real to derive certain identities, and then use analytic continuation to obtain them in general. Using Proposition 2.6, and rearranging various traces, we can derive the expressions

$$\lambda^2\mathrm{trace}\Big(G_x^{-1}Q_x\frac{dL_x}{dx}\Big) = \mathrm{trace}\Big((G_x^\dagger)^{-1}A - A + (G_x)^{-1}A^\dagger - A^\dagger\Big) \tag{6.15}$$

and likewise

$$\lambda^2\mathrm{trace}\Big(G_x^{-1}\frac{dQ_x}{dx}L_x\Big) = \mathrm{trace}\Big((G_x^\dagger)^{-1}A - A + (G_x)^{-1}A^\dagger - A^\dagger\Big), \tag{6.16}$$

and since $\frac{dG_x}{dx} = \lambda^2(\frac{dQ_x}{dx}L_x + Q_x\frac{dL_x}{dx})$, we deduce that

$$\begin{aligned}U(x,x) &= \frac{1}{2}\mathrm{trace}\Big(G_x^{-1}\frac{dG_x}{dx}\Big)\\ &= \frac{1}{2}\frac{d}{dx}\log\det G_x\\ &= \frac{1}{2}\frac{d}{dx}\log\det(I + \lambda^2\Gamma_{\phi_{(x)}}\Gamma^\dagger_{\phi_{(x)}}).\end{aligned}\tag{6.17}$$

This concludes the proof of the Lemma, hence of Theorem 6.1(ii) and Theorem 1.2(iii).

$\square$

We let $q \in C_0^\infty(\mathbf{R};\mathbf{C})$ and consider the Zakharov–Shabat system

$$\frac{d}{dx}\Psi(x;k) = \begin{bmatrix} -ik & q(x)\\ -\bar{q}(x) & ik\end{bmatrix}\Psi(x;k) \tag{6.18}$$

with $\Psi(x;k)$ a complex 2×2 matrix. We observe that this matrix is skew-symmetric with zero trace, so the norm of any solution is invariant under the evolution, as is the Wronskian of any pair of solutions; hence the fundamental solution matrix of this system belongs to $SU(2)$. We introduce the solutions $\Psi_+(x;k), \Psi_-(x;k)\in SU(2)$ such that

$$\Psi_+(x;k)\asymp\begin{bmatrix} e^{-ikx} & 0\\ 0 & e^{ikx}\end{bmatrix}\qquad (x\to\infty), \tag{6.19}$$

$$\Psi_-(x;k)\asymp\begin{bmatrix} e^{-ikx} & 0\\ 0 & e^{ikx}\end{bmatrix}\qquad (x\to-\infty); \tag{6.20}$$

then we introduce the scattering matrix $S(k)\in SU(2)$ such that $\Psi_-(x;k)=\Psi_+(x;k)S(k)$ and we write

$$S(k)=\begin{bmatrix}\alpha(k) & \hat\beta(k)\\ \beta(k) & -\hat\alpha(k)\end{bmatrix}. \tag{6.21}$$

Now suppose that α and β are analytic on the upper half-plane, and that α has zeros at κ_j. As in [6], we introduce the scattering data

$$\phi(x)=\sum_{j=1}^{n}\frac{\beta(\kappa_j)}{\alpha'(\kappa_j)}e^{i\kappa_j x}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\beta(k)}{\alpha(k)}e^{ikx}\,dk. \tag{6.22}$$

The sum contributes a function that decays exponentially as $x\to\infty$.

Proposition 6.4. *Let $(-A,B,C)$ realise the scattering data ϕ of the ZS system, suppose that the Gramians Q_x and L_x are Hilbert–Schmidt. Then the potential satisfies*

$$|q(x)|^2=\frac{1}{2}\frac{d^2}{dx^2}\log\det(I+\lambda^2\Gamma_{\phi_{(x)}}\Gamma^{\dagger}_{\phi_{(x)}}). \tag{6.23}$$

Lemma 6.5. *Let T be as in Lemma 6.2 and (6.2), and let*

$$\Psi(x;k)=\begin{bmatrix} ae^{ikx}\\ be^{-ikx}\end{bmatrix}+\int_x^{\infty}T(x,y)\begin{bmatrix} ae^{iky}\\ be^{-iky}\end{bmatrix}dy. \tag{6.24}$$

Then

$$-\frac{d^2}{dx^2}\Psi(x;k)+W(x)\Psi(x;k)=k^2\Psi(x;k) \tag{6.25}$$

where $W(x)=-2\frac{d}{dx}T(x,x)$.

Proof. One can follow the proof of Lemma 4.1 and deduce that

$$\frac{\partial^2}{\partial x^2}T(x,y)-\frac{\partial^2}{\partial y^2}T(x,y)=W(x)T(x,y). \tag{6.26}$$

Then one can verify the differential equation for $\Psi(x;k)$ by direct calculation.

From the original differential equation (6.18) we have

$$-\frac{d^2}{dx^2}\Psi(x;k) + \begin{bmatrix} -|q|^2 & q' \\ -\bar{q}' & -|q|^2 \end{bmatrix} \Psi(x;k) = k^2\Psi(x;k), \tag{6.27}$$

so by equating the matrix potential with $-2\frac{d}{dx}T(x,x)$, we obtain

$$\begin{bmatrix} -|q|^2 & q' \\ -\bar{q}' & -|q|^2 \end{bmatrix} = -2\frac{d}{dx}\begin{bmatrix} \bar{U}(x,x) & V(x,x) \\ -\bar{V}(x,x) & U(x,x) \end{bmatrix}. \tag{6.28}$$

□

Finally, we consider how the potential $q(x)$ evolves to $u(x,t)$ under the nonlinear Schrödinger equation. Suppose that

$$W_t(x;\zeta) = \begin{bmatrix} -i\zeta & u \\ -\bar{u} & i\zeta \end{bmatrix}, \qquad Z_t(x;\zeta) = \begin{bmatrix} -i|u|^2 + 2i\zeta^2 & -i\frac{\partial u}{\partial x} - 2u\zeta \\ -i\overline{\frac{\partial u}{\partial x}} + 2\bar{u}\zeta & i|u|^2 - 2i\zeta^2 \end{bmatrix}. \tag{6.29}$$

Proposition 6.6. *Suppose that u satisfies the nonlinear Schrödinger equation*

$$i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u. \tag{6.30}$$

(i) Then the pair of differential equations

$$\begin{cases} \frac{d}{dx}\Psi = W_t(x;\zeta)\Psi \\ \frac{d}{dt}\Psi = Z_t(x;\zeta)\Psi \end{cases} \tag{6.31}$$

gives a consistent system.
(ii) Let $\Psi_t(x,\zeta)$ be family of solutions to (6.31) with initial value $\Psi_t(0,\zeta) = \Psi_t$. Then the kernels

$$K_{t,x}(\kappa,k) = \frac{\langle J\Psi_t(x;\kappa), \Psi_t(x;k)\rangle}{\kappa - k} \tag{6.32}$$

satisfy (10^o) and (5^o); so $\frac{\partial}{\partial x}K_{t,x}$ and $\frac{\partial}{\partial t}K_{t,x}$ have finite rank.
(iii) The evolution of the potentials under the NLSE gives rise to a linear evolution on the scattering data.

Proof. This is similar to that of Theorem 5.6.

□

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